Analytical procedure for determining lengths from fractional fringes

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The development of stabilized multifrequency lasers makes fractional fringes an increasingly attractive technique for length measurement. Determination of an unknown length from the measured fractional fringes is aided by the development of analytical equations for the length and its uncertainty, and criteria are given for selecting the wavelengths.

Introduction

The technique of fractional fringes or exact fractions has been used for a variety of length measuring tasks since Benoit[1] applied it to the determination of the meter. However, the use of fractional fringes with optical interferometry has been restricted by the limited availability of well known and appropriately spaced wavelengths. The development of frequency stabilized ir lasers with a large number of well characterized wavelengths[2,3] and the possibility of frequency stabilized dye lasers should broaden the use of fractional fringes. In this paper we develop a set of analytical equations, amenable to automatic computation, that permit the length to be calculated as a series of successive approximations in terms of the wavelengths and the measured fractional fringes. Included are criteria for selecting the wavelengths, expressions for the uncertainty of the calculated value at each stage of the computation, and expressions for the length range over which the calculation will be valid.

The determination of an unknown length from fractional fringes works as follows. A set of measured fractional fringes is obtained by measuring the unknown length interferometrically with two or more wavelengths. Only one discrete set of equally spaced lengths will satisfy that set of measured fractional fringes. This is illustrated in Fig. 1 for two wavelengths $\lambda_2 = 0.8\lambda_1$, with measured fractional fringes $f_1 = 0.4$ and $f_2 = 0.5$. These fractional fringe values occur simultaneously in the figure at lengths marked A,B,C that are spaced apart by a repeat distance $4\lambda_1$, or $5\lambda_2$. The measured length could be any one of those or the infinite number of other lengths for which this combination of f_1 and f_2 will occur. However, if an initial estimate of the length L' is available with an uncertainty $\pm\Delta L$, the true length L must be between $L' - \Delta L$ and $L' + \Delta L$. If ΔL is less than half of the repeat distance only one of the points for which $f_1 = 0.4$ and $f_2 = 0.5$ will be between $L' - \Delta L$ and $L' + \Delta L$. In this case it is point B and $L = L' - 1.8\lambda_1$.

Basic Equations

In the following derivation an initial estimate of the length will be corrected to obtain a value of the measured length with a smaller uncertainty than the initial estimate. This process may have to be repeated to obtain the final desired uncertainty. The initial estimate is assumed available using some other measurement technique, e.g., a micrometer or meter stick, although, as will be discussed, in some cases even an initial estimate of the length is not required. The correction term will depend on the initial estimate of the length and the measured fractional fringes for two or more wavelengths. It will be expressed in terms of a synthetic or effective wavelength, obtained by taking combinations of the individual wavelengths. The calculation will be possible only if a synthetic wavelength can be obtained that is longer than the total uncertainty in the measured length.

Given a set of wavelengths λ_i, $i = 1,2,\ldots$, we can define a set of corresponding wavenumbers $k_i = 1/\lambda_i$. We can then write, for a length L,

$$L = (N_i + f_i)\lambda_i$$

or

$$Lk_i = N_i + f_i, \tag{1}$$

where N_i are the integral numbers of fringes, and the fractional fringes f_i satisfy $0 \le f_i < 1$. If we have an estimate L' of the length we can use it to determine a set of G_i and e_i from the equations

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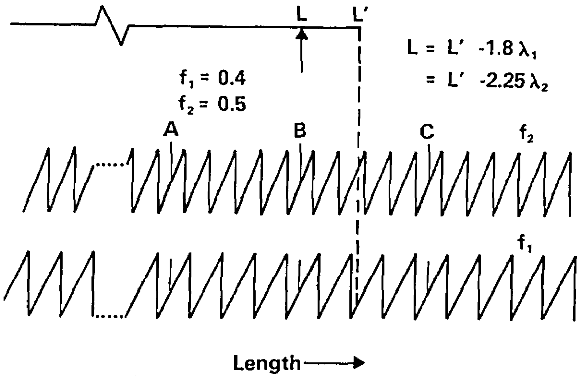

Fig. 1. Illustration of the use of fractional fringes f_1 and f_2 to determine a length L. The values of f_1, f_2, and the estimated length L' can be used to calculate an improved value for L.

$$L'k_i = G_i + e_i, \tag{2}$$

where G_i is the integral part of $L'k_i$, and e_i is the positive fractional part. If we can determine the difference between the true and estimated length, $\delta L = L - L'$, it can be used to correct the initial estimate of the length. δL can be obtained by subtracting Eq. (2) from Eq. (1),

$$\delta L k_i = N_i - G_i + f_i - e_i. \tag{3}$$

We can form linear combinations of the above equations,

$$\Sigma A_i \delta L k_i = \Sigma A_i (N_i - G_i + f_i - e_i),$$

where all sums are over i, and the A_i are zero, positive, or negative integers. Criteria for selecting the A_i will be developed later. Then,

$$\delta L = \frac{\Sigma A_i (N_i - G_i) + \Sigma A_i (f_i - e_i)}{\Sigma A_i k_i}.$$

Since A_i, N_i, and G_i are all integers, $\Sigma A_i (N_i - G_i) = I$, where I is some unknown integer. Therefore,

$$\delta L = \frac{I + \Sigma A_i (f_i - e_i)}{\Sigma A_i k_i} = [I + \Sigma A_i (f_i - e_i)]\lambda_s, \tag{4}$$

where $1/\Sigma A_i k_i = \lambda_s$ is the synthetic or effective wavelength for the beat frequency obtained by adding multiples of the different frequencies corresponding to the wavelengths λ_i. We can obtain a unique solution from Eq. (4) if we can determine what the value of I should be. This can be done if we select the A_i so that δL lies within a range spanned by this synthetic wavelength, i.e., $-\frac{1}{2}\lambda_s < \delta L < \frac{1}{2}\lambda_s$. If we know the uncertainty ΔL in our initial estimate of the length and are able to select the A_i such that the synthetic wavelength is long enough so that the above condition applies, the correct solution from Eq. (4) is that value of I for which

$$-\tfrac{1}{2} < I + \Sigma A_i (f_i - e_i) < \tfrac{1}{2}. \tag{5}$$

Only one value of I will satisfy this condition, and the solution will be unique.

Equation (4) and the condition imposed by Eq. (5) are all that are needed to determine δL, but they can be simplified for computational purposes by noting that for the numerical value of $I + \Sigma A_i (f_i - e_i)$ which satisfies Eq. (5),

$$I + \Sigma A_i (f_i - e_i) = F[\Sigma A_i (f_i - e_i)] \tag{5a}$$

or

$$= F[\Sigma A_i (f_i - e_i)] - 1, \tag{5b}$$

where $F[\Sigma A_i (f_i - e_i)]$ is the positive fractional part of $\Sigma A_i (f_i - e_i)$ and $0 \le F[\Sigma A_i (f_i - e_i)] < 1$. If $0 < F[\Sigma A_i (f_i - e_i)] < \frac{1}{2}$ Eq. (5a) is used, if $\frac{1}{2} < F[\Sigma A_i (f_i - e_i)] < 1$, Eq. (5b) is used. Similarly, the value of $F[\Sigma A_i (f_i - e_i)]$ can be written as

$$F[\Sigma A_i (f_i - e_i)] = F(\Sigma A_i f_i) - F(\Sigma A_i e_i)$$

or

$$= F(\Sigma A_i f_i) - F(\Sigma A_i e_i) + 1.$$

Equation (4) can then be rewritten as

$$\delta L = [I' + F(\Sigma A_i f_i) - F(\Sigma A_i e_i)]\lambda_s.$$

Equation (5) requires the term in brackets to be between $-\frac{1}{2}$ and $+\frac{1}{2}$ which will be the case only for $I' = -1$, 0, or 1.

The above expression for δL can be further simplified by taking linear combinations of Eq. (2),

$$\Sigma A_i e_i = \Sigma A_i L' k_i - \Sigma A_i G_i.$$

Since $\Sigma A_i G_i$ is an integer

$$F(\Sigma A_i e_i) = F(L' \Sigma A_i k_i)$$

$$= F(L'/\lambda_s).$$

Equation (4) can then be finally rewritten as

$$\delta L = [I' + F(\Sigma A_i f_i) - F(L'/\lambda_s)]\lambda_s, \tag{6}$$

where $I' = -1, 0$, or 1. The unique solution is the value of I' for which

$$-\tfrac{1}{2} < I' + F(\Sigma A_i f_i) - F(L'/\lambda_s) < \tfrac{1}{2}. \tag{7}$$

If the available wavelengths $\lambda_i = 1/k_i$ can be combined to form a synthetic wavelength $\lambda_s = 1/\Sigma A_i k_i$ that spans the possible range of δL, Eq. (6) can be used to obtain a corrected value of the length $L = L' + \delta L$. However, as discussed in the next section, the f_i and k_i are measured values, and their experimental uncertainties will introduce an uncertainty into the corrected length so that further corrections may be required. The experimental uncertainties also will further restrict the applicability of our solution.

Effects of Experimental Uncertainties

In general, if the k_i are corrected for the index of refraction of the medium they will be well known, and we will not explicitly discuss the effect of their uncertainties. It should be noted, however, that they could become important if a very long synthetic wavelength is generated so that $\Sigma A_i k_i$ becomes small enough to be comparable to the uncertainty in one of the k_i.

The uncertainties in the f_i will typically be 0.01 although they can easily be an order of magnitude smaller or larger depending on the particular experiment. We have

$$L = L' + \delta L.$$

Since δL depends on L' and the f_i, the uncertainty dL in the derived length will be, neglecting errors in the k_i,

$$dL = \frac{\partial L'}{\partial L'} dL' + \frac{\partial \delta L}{\partial L'} dL' + \Sigma \frac{\partial \delta L}{\partial f_i} df_i,$$

where dL' and df_i are the uncertainties in L' and the f_i. Now, taking linear combinations of Eqs. (1) and (2) we can write

$$\begin{aligned}\delta L &= [\Sigma A_i(N_i + f_i) - \Sigma A_i(G_i + e_i)]\lambda_s \\ &= [\Sigma A_i N_i + \Sigma A_i f_i - L'\Sigma A_i k_i]\lambda_s\end{aligned}$$

so

$$(\partial \delta L)/(\partial L') = -(\Sigma A_i k_i)\lambda_s = -1$$

and

$$(\partial \delta L)/(\partial f_i) = A_i \lambda_s.$$

Hence,

$$\begin{aligned}dL &= dL' - dL' + (\Sigma A_i df_i)\lambda_s \\ &= (\Sigma A_i df_i)\lambda_s.\end{aligned}$$

This demonstrates that the solution is self-correcting, i.e., the initial error in L' does not propagate through the solution but is canceled out, and the uncertainty of the corrected value of the length depends only on the uncertainties in the f_i. If all the f_i have the same uncertainty, $\pm\epsilon$, the corrected length will be uncertain by $\pm\lambda_s \epsilon \Sigma |A_i| = \pm\lambda_s \gamma$, where $\gamma = \Sigma |A_i| \epsilon$.

Not only will the uncertainties in the f_i determine the uncertainty in the corrected length, but they will further restrict the range over which Eq. (6) has a unique solution. Figure 2 shows schematically a length L' with the brackets indicating the uncertainty in that value of L'. The true length L, which can be anywhere between the brackets, is indicated as being near one extreme. A synthetic wavelength somewhat longer than the uncertainty in L' is shown below. Neglecting errors, that synthetic wavelength could be used to obtain a corrected length. However, as shown, the corrected length will have a possible error of $\pm\lambda_s\gamma$. If the true length is within $\lambda_s\gamma$ of either end of the synthetic wavelength in Fig. 2 the corrected length could appear to be beyond the end, as indicated by L_c. However, the calculated value will then be displaced by one synthetic wavelength to L_c' on Fig. 2. To avoid this error the uncertainty in L' must be less than the synthetic wavelength minus the uncertainty in the corrected value. That is, we can obtain a corrected length for a given set of A_i if

$$\frac{-(1 - 2\gamma)\lambda_s}{2} < \delta L < \frac{(1 - 2\gamma)\lambda_s}{2}. \qquad (8)$$

Obviously no solution can be obtained if $\gamma \geq \frac{1}{2}$. We will call $\pm[(1 - 2\gamma)\lambda_s]/2$ the useful range of a given synthetic wavelength.

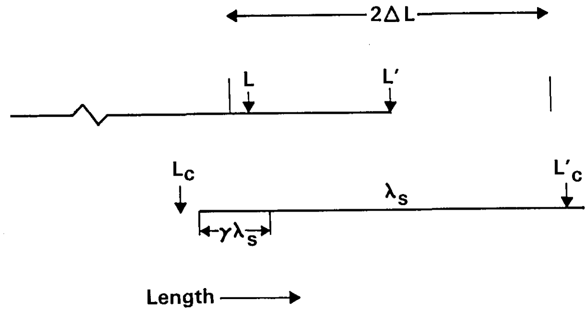

Fig. 2. Illustration of possible error of one synthetic wavelength due to uncertainty in the measured fringes. This error can be avoided by restricting the usable portion of the synthetic wavelength to $(1 - 2\gamma)\lambda_s$.

Use of the Equations

The general procedure is to select, if possible, a combination of wavenumbers such that the useful range of the synthetic wavelength spans the uncertainty in the initial determination of length, and Eq. (8) is satisfied. A corrected length is calculated using Eq. (6) with a resultant uncertainty of $\pm\gamma\lambda_s$. A new combination of wavenumbers, $\Sigma A_i' k_i$, is picked with a shorter synthetic wavelength such that

$$\frac{\epsilon \Sigma |A_i|}{|\Sigma A_i k_i|} < \frac{1 - 2\epsilon \Sigma |A_i'|}{|\Sigma A_i' k_i|},$$

and Eq. (6) is again applied to obtain a second correction to the length. This process is repeated using progressively shorter synthetic wavelengths until the uncertainty is less than a fringe of one of the λ_i. The corrected length will then have an error determined by only one of the measured partial fringes, i.e., the integral number of fringes or order number is known exactly.

It should be noted that this technique can be used without an initial estimate of the length if a synthetic wavelength long enough to span the desired measurement range can be generated. In this case $L' = 0$ can be used in Eq. (6), and δL is the first approximation to the length. It is also worth noting that if L is known to be positive and sufficiently far from zero so that experimental uncertainties cannot make it appear negative, the range of the approximation can be doubled. This is also true if L is known to be negative and sufficiently removed from zero. In these two cases Eq. (8) is modified to

$$\gamma\lambda_s < L < (1 - \gamma)\lambda_s \qquad (L > 0)$$

or

$$-(1 - \gamma)\lambda_s < L < -\gamma\lambda_s \qquad (L < 0),$$

and a unique solution can be obtained from

$$L = F(\Sigma A_i f_i)\lambda_s \qquad (L > 0)$$

or

$$L = [F(\Sigma A_i f_i) - 1]\lambda_s \qquad (L < 0).$$

Selection of the proper combination of wavenumbers to generate the desired synthetic wavelengths can present a problem. If wavelengths can be selected at will, as could be the case with microwave or acoustical interferometers, arbitrarily long effective wavelengths can be generated by taking simple differences between pairs of wavelengths, although at some point uncertainties in the measured relative values of the wavelengths will start to contribute errors. If a limited number of fixed wavelengths are available, there will be a limit to the range of synthetic wavelengths for which Eq. (8) can be satisfied, and the choice of integral coefficients A_i to obtain the desired synthetic wavelengths may involve some guessing and trying. Once a set of synthetic wavelengths are generated, they can be used to determine any length for which Eq. (8) is satisfied.

In the preceding derivations it has been assumed that the experimental uncertainties are known. They can of course only be estimated. Since an error of one or more synthetic wavelengths will occur in the corrected length if the conditions of Eq. (8) are not met, it is wise to be conservative in estimating experimental uncertainties and check the final corrected length if possible. The latter could be done by comparing a final calculated partial fringe with the measured fringes for all wavelengths.

Example

Development of the computational technique described in this paper was initiated by the use of pulsed ultrasonic[4] and CO_2 laser interferometers[5] for the accurate determinations of liquid column heights in manometry. Both of these systems used cumulative fringe counting which has to proceed very slowly when mercury is used because of ripples on the moving mercury surface. The fractional fringe technique is obviously well suited for use with the ultrasonic interferometer since it is a relatively simple matter to use the interferometer with several frequencies. These can be chosen at will so that simple difference, between wavenumbers will give any desired synthetic wavelength. The CO_2 laser interferometer can also be used with this technique. Although the wavelength cannot be arbitrarily set, the CO_2 laser will generate a large number of regularly but not quite evenly spaced wavelengths in two bands centered near 9.4 μm and 10.4 μm. About seventy or eighty of these wavelengths can be obtained with a reproducibility of parts in 10^{10} from a stabilized laser. These wavelengths have been measured with a relative uncertainty of less than 3 parts in 10^{10} and an absolute uncertainty of about 4 parts in 10^9.[6] Thus a stabilized CO_2 laser is an excellent length standard, and the spacing of wavelengths permits the generation of a wide range of synthetic wavelengths.

An attempt has not been made to determine the largest usable synthetic wavelength obtainable using a CO_2 laser, but the following examples illustrate the possibilities. Taking differences between the $R(24)$, $R(26)$, $R(28)$, and $R(30)$ lines in the 9.4-μm band gives a synthetic wavelength of 103 m. The small uncertainties in the measured differences between the individual wavelengths result in an uncertainty in this synthetic wavelength of about 0.4%. However, it is not apparent that an object of this length would be stable to a part in 10^9 long enough to measure the fractional fringes with a precision of 0.01.

More realistically, differences between the $R(24)$, $R(26)$, and $R(28)$ lines give a synthetic wavelength of 380.65 mm. Two more lines spaced further apart would be required to generate a series of synthetic wavelengths such that the uncertainty in the measured length is reduced below one fringe (assuming an uncertainty of ± 0.01 in the measured fractional fringes). If a shorter range is required fewer wavelengths are needed. For example, assume the following three wavelengths in the 9.4-μm band are used:

$$\lambda_1 = R(28) = 9.22953010\ \mu\text{m};$$

$$\lambda_2 = R(24) = 9.24994570\ \mu\text{m};$$

$$\lambda_3 = P(32) = 9.65741651\ \mu\text{m}.$$

Then, assuming an uncertainty in the measured fringes of ± 0.01, the following A_i will generate a set of synthetic wavelengths that can be used to reduce the uncertainty in the length below one fringe.

$$A_1 = 1, A_2 = -1, A_3 = 0 \qquad \lambda_{s1} = 4.1817\ \text{mm};$$

$$A_1 = 1, A_2 = 0, A_3 = -1 \qquad \lambda_{s2} = 0.2083109\ \text{m};$$

$$A_1 = -1, A_2 = 0, A_3 = 2 \qquad \lambda_{s3} = 0.0101269060\ \text{mm}.$$

It should be noted that while shorter synthetic wavelengths can be generated by multiplying all the A_i by a common integer nothing is gained since the relative error is correspondingly increased, and the absolute error remains constant. As an example, if $A_i' = 2A_i$, then $\lambda_s' = \frac{1}{2}\lambda_s$. But we will have $\gamma' = 2\gamma$ so $\gamma'\lambda_s' = \gamma\lambda_s$.

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